

AFRL-AFOSR-UK-TR-2013-0035



**Development of a Class of Smoothness-Increasing-Accuracy-
Conserving (SIAC) Methods for Post-Processing
Discontinuous Galerkin Solutions**

Jennifer K. Ryan

**Technische Universiteit Delft
Delft Institute of Applied Mathematics
Mekelweg 4
Delft 2628 CN
NETHERLANDS**

EOARD Grant 09-3055

Report Date: July 2013

Final Report from 01 April 2009 to 31 March 2013

Distribution Statement A: Approved for public release distribution is unlimited.

**Air Force Research Laboratory
Air Force Office of Scientific Research
European Office of Aerospace Research and Development
Unit 4515 Box 14, APO AE 09421**

REPORT DOCUMENTATION PAGE				Form Approved OMB No. 0704-0188	
<small>Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing the burden, to Department of Defense, Washington Headquarters Services, Directorate for Information Operations and Reports (0704-0188), 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.</small> PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.					
1. REPORT DATE (DD-MM-YYYY) 30 July 2013		2. REPORT TYPE Final Report		3. DATES COVERED (From – To) 1 April 2009 – 31 March 2013	
4. TITLE AND SUBTITLE Development of a Class of Smoothness-Increasing-Accuracy-Conserving (SIAC) Methods for Post-Processing Discontinuous Galerkin Solutions				5a. CONTRACT NUMBER FA8655-09-1-3055	
				5b. GRANT NUMBER Grant 09-3055	
				5c. PROGRAM ELEMENT NUMBER 61102F	
				5d. PROJECT NUMBER	
6. AUTHOR(S) Jennifer K. Ryan				5d. TASK NUMBER	
				5e. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Technische Universiteit Delft Delft Institute of Applied Mathematics Mekelweg 4 Delft 2628 CN NETHERLANDS				8. PERFORMING ORGANIZATION REPORT NUMBER N/A	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) EOARD Unit 4515 APO AE 09421-4515				10. SPONSOR/MONITOR'S ACRONYM(S) AFRL/AFOSR/IOE (EOARD)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S) AFRL-AFOSR-UK-TR-2013-0035	
12. DISTRIBUTION/AVAILABILITY STATEMENT Distribution A: Approved for public release; distribution is unlimited.					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT Although discontinuous and continuous Galerkin methods have advantages mathematically and computationally, they suffer from one feature that can in turn become a disadvantage - they do not require high levels of smoothness at the element boundaries. Lack of smoothness across elements can hamper simulation post processing like feature extraction and visualization. The purpose of this proposal is to develop smoothness-increasing accuracy-conserving filters that respect the mathematical properties of the data while providing levels of smoothness so that commonly used visualization tools can be used appropriately, accurately, and efficiently. The goals of this effort are to define, investigate, and address the technical obstacles inherent in visualization of data derived from high-order discontinuous Galerkin methods and to provide robust and easy to use algorithms to overcome the difficulties that arise due to lack of smoothness. In particular, we propose to contribute both mathematically and algorithmically to the class of smoothness increasing and accuracy-conserving (SIAC) methods and to provide a robust and freely available software solution to the high-order simulation community.					
15. SUBJECT TERMS EOARD, Galerkin methods, Discontinuous methods, SIAC, smoothness algorithms					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT SAR	18. NUMBER OF PAGES 8	19a. NAME OF RESPONSIBLE PERSON Gregg Abate
a. REPORT UNCLAS	b. ABSTRACT UNCLAS	c. THIS PAGE UNCLAS			19b. TELEPHONE NUMBER <i>(Include area code)</i> +44 (0)1895 616021

DEVELOPMENT OF A CLASS OF SMOOTHNESS-INCREASING ACCURACY-CONSERVING (SIAC) METHODS FOR POST-PROCESSING DISCONTINUOUS GALERKIN SOLUTIONS

EOARD/AFOSR FA86550913055

Jennifer K. Ryan
Delft Institute of Applied Mathematics
Delft University of Technology

Abstract

Although discontinuous and continuous Galerkin methods have advantages mathematically and computationally, they suffer from one feature that can in turn become a disadvantage - they do not require high levels of smoothness at the element boundaries. Lack of smoothness across elements can hamper simulation post-processing like feature extraction and visualization. The purpose of this proposal is to develop smoothness-increasing accuracy-conserving filters that respect the mathematical properties of the data while providing levels of smoothness so that commonly used visualization tools can be used appropriately, accurately, and efficiently. The goals of this effort are to define, investigate, and address the technical obstacles inherent in visualization of data derived from high-order discontinuous Galerkin methods and to provide robust and easy to use algorithms to overcome the difficulties that arise due to lack of smoothness. In particular, we propose to contribute both mathematically and algorithmically to the class of smoothness-increasing and accuracy-conserving (SIAC) methods and to provide a robust and freely available software solution to the high-order simulation community.

This work is done in active collaboration with Dr. Robert M. Kirby at the University of Utah, who is sponsored by the Air Force Office of Scientific Research, Air Force Material Command, USAF, under grant number FA9550-08-1-0156 and currently under grant number FA9550-12-1-0428.

Status/Progress

AFOSR funding to support this research was obtained in April 2009. This funding is currently used to support Mr. Xiaozhou Li (December 2010 – current) and Ms. Mathea Vuik (October 2012 – current), PhD researchers in Numerical Analysis at Delft University of Technology. Previous funds supported Ms. Paulien van Slingerland (April 2009 – June 2010). Ms. van Slingerland successfully defended her thesis in June of 2013 and Mr. Li is expected to defend his thesis in December 2014. During the Spring of 2012, Mr. Li worked with Dr. Kirby and his former PhD student Hanieh Mirzaee at the University of Utah.

This funding has allowed us to make several contributions over the lifetime of this grant:

- In the first year of the project, the focus was on creating a more effective one-sided post-processing technique that allows for maintaining the appropriate boundary values and producing errors that were of the same magnitude as in the interior. By modifying the filter based upon the evaluation point, we overcame the decrease in accuracy at the boundaries. This position-dependent Smoothness-Increasing Accuracy-Conserving (SIAC) filter for enhancing discontinuous

Galerkin solutions easily switches between one-sided post-processing to handle boundaries or discontinuities and symmetric post-processing for smooth regions. Different filtering kernels are used for different domain regions. The improvements to the one-sided kernel are accomplished by combining previous concepts by the PI used in one-sided post-processing for DG solutions with those from spectral methods and finite difference methods to improve the one-sided filter. This work resulted two papers [15,18] and has been presented in various arenas [22,24-26,30-32]. We then focused on the application of the new position-dependent SIAC filter to streamline visualization [18].

- In the second year of the grant, the theoretical extension of the symmetric kernel to nonlinear hyperbolic equations [19] was performed. The extension to nonlinear hyperbolic equations is possible provided the derivative of the flux with respect to u is bounded. In this case, it is possible to improve the order of the accuracy to $(2k+m)$, where m depends upon the numerical flux. This was presented at the U.S. National Congress on Computational Mechanics [29] as well as the European Conference on Numerical Mathematics and Advanced Applications [23].
- The third year of the grant concentrated on theoretical and computational extensions that are more useful for dealing with various visualization applications (streamlines, streaklines and isosurfaces). For the theoretical extensions, pointwise error estimates demonstrating that higher-order accuracy of order $2k+2-[d/2]$ is indeed achieved in the L^∞ -norm, where d is the dimension and k is the highest degree polynomial used in the approximation [18]. Theoretical results extending the current L^2 -error estimates to the *entire* domain were also done. This was a significant extension as pointwise error estimates will be more useful for quantifying errors in isosurface extraction. Further, together with Mike Kirby, the filter was demonstrated to be computationally viable for structured triangular meshes [17]. This work was presented at the SIAM Conference on Computational Science and Engineering [27-28]. Further, in the third year there were significant benefits from the collaboration that combines both mathematics and computation. We were able to make significant strides forward in both the theoretical and computational viability of the SIAC filter for applications. The theoretical results obtained include the extension of the L^2 -error estimates for: variable coefficient hyperbolic equations for DG solutions over a structured triangular mesh [17], as well as extending the theoretical results to adaptive meshes [20]. Combining these proofs with the computational results that demonstrate the efficiency of the filter [16] as well as applicability with GPU computing [20] allowed us to piece together the components necessary to recover higher-order accuracy for unstructured triangular meshes in the remaining duration of the grant and give insight into possible modification of the filter. These results were also presented in various arenas [2-4,12-14,21-22].
- In the last year of this grant, the focus has been on using the mathematical analysis and creating more computationally efficient solutions for various geometries, including unstructured triangular meshes. These explorations continue to focus on post-processing near boundaries, as the insight gained is more useful for application to unstructured triangular meshes. The main challenge for the one-sided post-processor in [15] is that for higher-order approximations, the number of B-splines needed is excessive and that increased precision is required. The excessive number of B-splines creates three problems: it increases the constant in the error term significantly, makes for a large condition number in the matrix that determines the kernel coefficients, and requires a larger support – causing the

filter to become more global. The combination of these makes for longer computations and results in round-off errors when computing using double precision. The major finding was that reducing the number of central B-Splines back to $2k+1$, shifting the kernel nodes and using one general B-spline improved the computational efficiency significantly. This modifies the shape of the kernel so that actual boundary information has more weight than for the previous kernel. This makes the kernel more computationally efficient while improving the smoothness of the solution (see Figure 1). This work has resulted in two papers [6,7] as well as one in preparation [5]. It has been presented in a variety of venues [1-4,8,11].

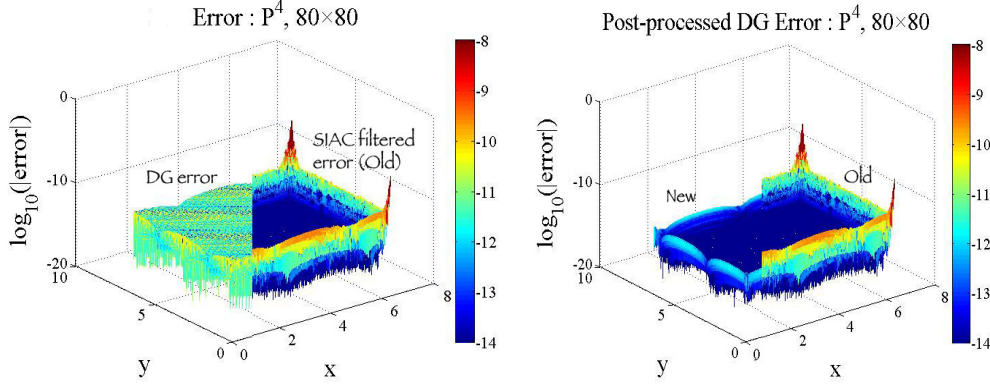


Figure 1: Left: A comparison of the errors in the DG approximation to the SIAC filtered approximation using the boundary method introduced in [15]. Higher computational costs are encountered along with round-off error at the boundaries. Right: A comparison of the SIAC filtered errors with the modified filter introduced in [6] and the filter used in [15]. The new boundary filter is more computationally efficient and results in less round-off error.

The results of this work are currently being used to improve the streamline and streakline integration process (see Figure 2 and [5]). Investigating streamlining requires two approaches: (1) Filtering the entire field and then performing the streamline integration; and (2) Using one-sided filtering along the streamline *during* the streamline integration process. The latter approach leads to more questions about the content of the errors, including which are from the filter and which are from the time-stepping. In order to reduce the errors from the time-stepping, we have created a one-sided derivative filter so that backwards differentiation formulas (BDF) can be used. Regardless of the approach, the SIAC filtered DG solution always does as well as the DG solution, and sometimes better at obtaining the correct streamline (see Figure 2).

Further to this, the first proof that the SIAC filter is indeed accuracy conserving,

$$||u(x,t)-u_h^*(x,t)|| \leq ||u(x,t)-u_h(x,t)||,$$

was obtained. In this estimate, $u(x,t)$ is the exact solution, $u_h(x,t)$ is the approximation solution obtained through use of the discontinuous Galerkin (DG) discretization and $u_h^*(x,t)$ is the SIAC filtered DG solution [6]. This is a very important result that combines the knowledge gained throughout the grant and helps to give insight into the role of the B-Splines and kernel shape determine the accuracy-conserving and accuracy-increasing properties.

Lastly, related to boundary issues and computational performance is the issue of geometry. This required performing a study of the kernel scaling for various mesh types. These results showed that the kernel scaling is the optimal point where superconvergence and improved errors align. This allowed us to properly scale the kernel for use on unstructured triangular meshes [9] as well as

structured tetrahedral meshes [10] in order to obtain reduced errors and a smoother solution. More importantly, the results from the study of the mesh scaling give are a crucial step in our future work. Specifically, it is clear that the mesh scaling for the kernel filter will be chosen not only on the polynomial degree, but also based on the mesh resolution. It also gives insight into how to modify the kernel function based on the mesh.

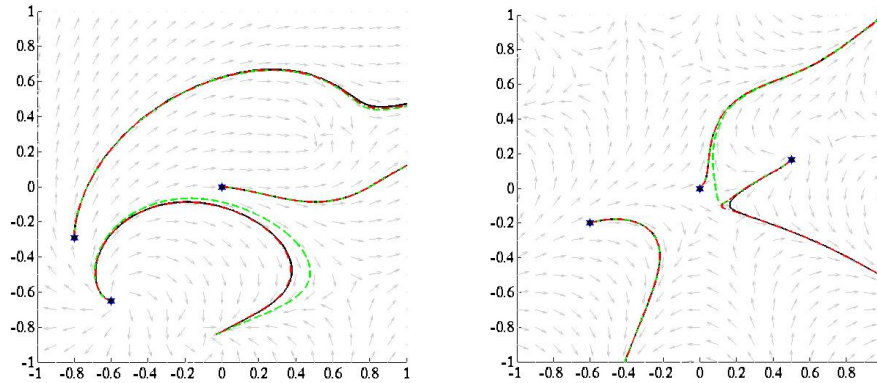


Figure 2: Left: Filtering the entire field and then calculating the streamline. Right: Filtering the streamline during the streamline calculation. Both streamline calculations use BDF. Black streamlines were created based upon integration on the continuous field, green streamlines were created based upon integration on the discontinuous Galerkin field and red streamlines denote the DG filtered field.

Most Significant Accomplishments

Over the period of the grant, the most significant accomplishments are:

- (1) The establishment through empirical study of a position-dependent smoothness-increasing accuracy-conserving (SIAC) filter that easily switches between a symmetric kernel for use in the domain interior to a one-sided kernel for use near boundaries [6];
- (2) The establishment of various error estimates:
 - a. The first L^∞ -error estimates for this position-dependent filter for hyperbolic equations [18];
 - b. L^2 -estimates that establish that SIAC filtering methodology can be applied to structured triangular meshes [17];
 - c. The establishment of L^2 -estimates for nonlinear hyperbolic conservation laws with bounded flux function [19];
 - d. The establishment of the effectiveness of the SIAC filter on adaptive meshes [20];
 - e. The establishment of the accuracy-conserving nature of the SIAC filter [6] and
- (3) The computational efficiency of the filter for one-sided post-processing [6], parallel and GPU computing [16,20] as well as for triangular, adaptive, and tetrahedral meshes [9,10,17,20].
- (4) In the last year of funding, the most significant accomplishments were carried out: That of defining the necessary geometric conditions for appropriate filtering and computationally efficient filtering and proof that the SIAC filter is indeed accuracy-conserving [5-7].

Acknowledgment/Disclaimer

This work is sponsored by the Air Force Office of Scientific Research, Air Force Material Command, USAF, under grant/contract number FA8655-09-1-3055. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Office of Scientific Research or the U.S. Government.

Personnel Supported During Duration of Grant

Xiaozhou Li	Ph.D. student, Delft University of Technology
Paulien van Slingerland	Ph.D. student, Delft University of Technology
Mathea Vuik	Ph.D. student, Delft University of Technology
Jennifer K. Ryan	Assistant Professor, Delft University of Technology

References

1. Xiaozhou Li, Minisymposia presentation, **Mathematics of Finite Elements and Applications (MAFELAP 2013)**, Brunel University, Uxbridge, United Kingdom. June 11-14, 2013.
2. Jennifer Ryan, *OCCAM seminar*, **University of Oxford**, Oxford, United Kingdom. May 9, 2013.
3. Jennifer Ryan, *Scientific Computing Seminar*, **University of Cambridge**, Cambridge, United Kingdom. April 17, 2013.
4. J.K. Ryan, **Advances in Computational Mechanics, A Conference Celebrating the 70th Birthday of Thomas J.R. Hughes**, San Diego, California. February 24-27, 2013.
5. X. Li, J.K. Ryan, R.M. Kirby, *One-Sided SIAC Filtering for Streamline with BDF Time Integrator*. Preprint, 2013.
6. X. Li, J.K. Ryan, R.M. Kirby, C. Vuik, *Computationally Efficient position-dependent filtering: The Uniform Mesh Case*. Submitted to **SIAM Journal on Scientific Computing**, 2013.
7. X. Li, J.K. Ryan, R.M. Kirby, C. Vuik, *Computationally Efficient position-dependent filtering: The Non-uniform Mesh Case*. Preprint, 2013.
8. X. Li, *Computationally Efficient position-dependent filtering for discontinuous Galerkin solutions*, Computer Science Seminar, **University of Utah**, April 9, 2012.
9. H. Mirzaee, J. King, J.K. Ryan, and R.M. Kirby, *Smoothness-Increasing Accuracy-Conserving (SIAC) filtering for discontinuous Galerkin solutions over unstructured meshes*, **SIAM Journal on Scientific Computing**, 35(2013), pp. A212-A230.
10. H. Mirzaee, J. King, J.K. Ryan, and R.M. Kirby, *Smoothness-Increasing Accuracy-Conserving (SIAC) Filters for Discontinuous Galerkin Solutions: Application to Structured Tetrahedral Meshes*, **Journal of Scientific Computing**, accepted.
11. J.K. Ryan, **International Conference on Applied Mathematics: Modeling, Analysis & Computation**, City University of Hong Kong, Hong Kong. May 28 – June 1, 2012.
12. J.K. Ryan, *Applications of Superconvergence*, Numerical Analysis Seminar, **Xiamen University**, Xiamen, Fujian, China. June 14, 2012.
13. J.K. Ryan, Numerical Analysis Seminar, **Courant Institute, New York University**, New York, NY, U.S.A. April 13, 2012.
14. J.K. Ryan, Scientific Computing Seminar, **Brown University**, Providence, RI, U.S.A. April 20, 2012.
15. P. van Slingerland, J.K. Ryan, C. Vuik, *Position-dependent smoothness-increasing accuracy-conserving (SIAC) filtering for improving discontinuous Galerkin solutions*, **SIAM Journal on Scientific Computing**, 33 (2011), pp.802-825.
16. Hanieh Mirzaee, Jennifer K. Ryan and Robert M. Kirby, *Efficient Implementations of Smoothness-Increasing Accuracy-Conserving (SIAC) Filters for Discontinuous Galerkin*

- Solutions*, **Journal of Scientific Computing**, 52(2012), pp. 85-112.
17. H. Mirzaee, L. Ji, J.K. Ryan and R.M. Kirby, *Smoothness-Increasing Accuracy Converging (SIAC) Post-Processing for Discontinuous Galerkin Solutions Over Structured Triangular Meshes*, **SIAM Journal on Numerical Analysis**, 49 (2011), 1899-1920.
 18. L. Ji, P. van Slingerland, J.K. Ryan, C. Vuik, *Superconvergent error estimates for position-dependent smoothness-increasing accuracy-conserving filtering for DG solution*, **Mathematics of Computation**, accepted.
 19. L. Ji, Y. Xu, J.K. Ryan, *Negative-order norm error estimates for nonlinear hyperbolic equations*, submitted to **Journal of Scientific Computing**, 54(2013), 269-310.
 20. J. King, H. Mirzaee, J.K. Ryan, and R.M. Kirby, *Smoothness-Increasing Accuracy-Conserving (SIAC) Filtering for discontinuous Galerkin Solutions: Improved Errors Versus Higher-Order Accuracy*. **Journal of Scientific Computing**, 53 (2012), pp. 129-149.
 21. J.K. Ryan, **SCPDE11 The 4th International Conference on Scientific Computing and Partial Differential Equations**, Hong Kong Baptist University. Hong Kong. December 5-9, 2011.
 22. J.K. Ryan, *Extracting superconvergence for discontinuous Galerkin solutions*, **Applied Mathematics Seminar**, University of East Anglia, Norwich, United Kingdom. October 31, 2011.
 23. L. Ji, Y. Xu, J.K. Ryan, *Superconvergence of discontinuous Galerkin methods for nonlinear hyperbolic conservation laws with smooth solutions*, Mini-symposia presentation at the **European Numerical Mathematics and Advanced Applications**, September 5-9, 2011, Leicester, United Kingdom.
 24. J.K. Ryan, *Position-Dependent Smoothness-Increasing Accuracy-Conserving (SIAC) Filtering for Discontinuous Galerkin Solutions*, **International Conference in Honor of Saul Abarbanel's 80th Birthday**, Tel Aviv University. Tel Aviv, Israel. June 28-29, 2011.
 25. J.K. Ryan, *Smoothness-Increasing Accuracy-Conserving (SIAC) Filtering: A component-by-component construction for equations with stiff source terms*, **Numerical Methods for Hyperbolic Equations Theory and Applications**. Santiago de Compostela, Spain. July 4-8, 2011.
 26. J.K. Ryan, **Applied Mathematics Seminar**, University of Nottingham, Nottingham, United Kingdom. April 7, 2011.
 27. Jennifer K. Ryan, Contributed talk, **SIAM Conference on Computational Science and Engineering**, February 28 – March 4, 2011, Reno, NV, U.S.A.
 28. Jennifer K. Ryan, Poster presentation, **SIAM Conference on Computational Science and Engineering**, February 28 – March 4, 2011, Reno, NV, U.S.A.
 29. L. Ji, *Superconvergence Extraction for Nonlinear Hyperbolic Equations*, **U.S. National Congress on Computational Mechanics (USNCCM 2011)**. Minneapolis, MN. July 25-29, 2011.
 30. P. van Slingerland, J.K. Ryan, C. Vuik, *Smoothness Increasing Accuracy Conserving Filtering Applied to Streamline Visualisation of DG Solutions*, **Technical Report 09-06**, Delft University of Technology, Delft, Netherlands, 2009.
 31. P. van Slingerland, *Numerical Analysis Seminar*, **Delft University of Technology**. Delft, Netherlands. April 22, 2010.
 32. P. van Slingerland, **Werkgemeenschap Scientific Computing Spring Meeting**. Antwerp, Belgium. May 3, 2010.